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Reconsideration of intermittent synchronization in coupled chaotic pendula

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We reinvestigate the *intermittent synchronization* phenomenon of coupled chaotic pendula that has recently been a controversy. We propose a simple numerical scheme by which one can easily determine whether the observed synchronization is a numerical artifact of computer analysis or not. By using this scheme, for certain coupling strength regime, we find that the average time taken for synchronization *linearly* depends on the precision of calculations. According to Longa *et al.*'s criterion for synchronization, this implies that the observed synchronization is genuine.

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Synchronization of coupled chaotic systems is one of the most intriguing aspects of chaotic dynamics. It has been extensively studied because of its potential importance in practical applications, e.g., secure communications, nonlinear dynamical model verification [1], etc.

Since the pioneering works [2] of Fujisaka and Yamada, Afraimovich *et al.*, and Pecora and Carroll, various synchronization schemes have been proposed. In some of those, there have been controversies on *whether the observed synchronization is accidental or genuine, i.e., whether it is an artifact of finite precision of numerical calculations or not*. The first of those controversies was raised by Maritan and Banavar (MB) [3]. They mistakenly claimed that two identical logistic maps coupled by common noise can be synchronized for appropriate coupling strengths. Pikovsky [4] pointed out that what they observed was an accidental synchronization, since the maximal Lyapunov exponent of the system is positive. Later, Longa *et al.* [5] explicitly showed that MB's synchronization is indeed a numerical artifact. They proposed a very useful criterion for chaos synchronization, which is that the average number of iterations taken for genuine synchronization *linearly* depends on the precision of calculations, whereas the accidental synchronization *exponentially* depends on the precision of calculations.

Recently, Baker, Blackburn, and Smith (BBS) [6] reported another controversial model for chaos synchronization. They studied chaotic flows of unidirectionally coupled pendula, which also serve as a model for the Josephson junction. Their claim was that the apparent complete synchronization of the system is a numerical artifact of computer analysis and that *intermittent synchronization is a plausible behavior.* In their numerical study, they randomly scrambled the nth decimal digits in the variable to prevent accidental synchronization, which is effectively equivalent to the addition of small noise. After BBS's report, Grassberger [7] commented that there is a genuine transition threshold coupling strength, at which the maximal Lyapunov exponent of the linearized difference motion between the master and slave system becomes 0, and that the intermittent synchronization BBS observed is nothing other than *on-off intermittency* [8] near the synchronization threshold [9]. In another comment, Muruganandam *et al.* [10] claimed that there exists some specific ranges of coupling strength for which persistent synchronization can occur, by numerical analysis of the conditional Lyapunov exponent. They also claimed that the conditional Lyapunov exponent plays an important role in distinguishing between intermittent and permanent synchronization. However, as BBS replied in [11], quantifying the condition for synchronization is a very subtle subject. BBS tested several possible candidates for a measure of the condition of synchronization. Among them are included Lyapunov function and the largest eigenvalue of the Jacobian of the flow. However, it has been reported theoretically, and observed in numerical and physical experiments, that a simple calculation of these quantities is insufficient for the prediction of synchronization [12,13].

Our motivation for this study is to explicitly clarify the controversy on intermittent synchronization. In this Rapid Communication, we propose a simple numerical scheme to determine whether the observed synchronization is accidental or genuine. With the help of this scheme, we are able to investigate the difference dynamics of unidirectionally coupled pendula up to the 10^{-308} order.

The motion of the master and the slave pendula are described in a dimensionless form by the usual nonautonomous expression in the angular coordinates θ_m, θ_s:

$$\ddot{\theta}_m + \gamma\dot{\theta}_m + \sin\theta_m = \Gamma_0\cos(\Omega t), \tag{1}$$

$$\ddot{\theta}_s + \gamma\dot{\theta}_s + \sin\theta_s = \Gamma_0\cos(\Omega t) + c(\sin\theta_s - \sin\theta_m),$$

where time t has been normalized in the unit of ω_0^{-1}, ω_0 being the small-angle resonant frequency of the pendulum, γ is the damping coefficient, Γ_0 is the amplitude of modulations normalized by the pendulum critical torque mgl, the drive frequency Ω is expressed in units of ω_0, and c is the coupling strength. These nonautonomous equations can be written as equivalent sets of autonomous coupled first-order differential equations as follows:

$$\dot{\theta}_m = \omega_m,$$

$$\dot{\omega}_m = -\gamma\omega_m - \sin\theta_m + \Gamma_0\cos\phi_m, \tag{2}$$

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$$\dot{\phi}_m = \Omega, \quad \text{and}$$

$$\dot{\theta}_s = \omega_s,$$

$$\dot{\omega}_s = -\gamma\omega_s - \sin\theta_s + \Gamma_0 \cos\phi_s + c(\sin\theta_s - \sin\theta_m), \quad (3)$$

$$\dot{\phi}_s = \Omega.$$

All the results that are presented in this paper were computed with the same parameter values that were used in BBS's study, i.e., $\gamma = 0.2$, $\Gamma_0 = 1.2$, and $\Omega = 0.5$. Both the master and the slave pendula are in a chaotic state in those parameter values. Numerical results are obtained with a fourth-order Runge-Kutta routine with time grids of $0.001(2\pi/\Omega)$. When synchronization is achieved, the strange attractor for the slave exactly coincides with that of the master, and the synchronization manifold M can be represented as $(\theta_m = \theta_s, \omega_m = \omega_s, \phi_m = \phi_s)$. In the absence of synchronization, the Poincaré points move on the two attractors in an uncorrelated fashion. To measure the quality of synchronization, we used the distance between a point on the master attractor and its corresponding point on the slave attractor in phase space, such as $\eta = \{(\theta_s - \theta_m)^2 + (\dot{\theta}_s - \dot{\theta}_m)\}^{1/2}$.

The conventional procedure for studies of numerical synchronization of coupled flows is as follows: (i) Integrate six-dimensional differential equations [Eqs. (2) and (3)] with fourth order Runge-Kutta algorithm; (ii) Calculate η to determine the degree of synchronization. A drawback of this conventional procedure is the phenomenon of hard locking when η is within the precision limit of the calculation. In order to avoid this difficulty, a small noise in the order of the precision of the computer, i.e., seven digits for single precision or 15 digits for double precision, is usually added to each variable at every Runge-Kutta step to perform the numerical study. However, the true dynamics of the system can be perturbed by this small added noise.

Here, we propose to study an equivalent set of equations, by transforming the slave equations to the equations of difference motion between the master and the slave to circumvent the hard locking problem, instead of adding noise. The transformation of the variables, $\theta = \theta_m - \theta_s$, $\omega = \omega_m - \omega_s$, and $\phi = \phi_m - \phi_s$ leads to the equations of difference motion between the master and the slave as follows:

$$\dot{\theta} = \omega,$$

$$\dot{\omega} = -\gamma\omega - 2(1-c)\sin\left(\frac{\theta}{2}\right)\cos\left(\theta_m - \frac{\theta}{2}\right), \quad (4)$$

$$\dot{\phi} = 0.$$

Then, the synchronization manifold M corresponds to $(\theta, \omega, \phi) = (0,0,0)$ and the coordinates are taken in a direction orthogonal to M. Notice that the transformation from the variables $(\theta_m, \omega_m, \phi_m, \theta_s, \omega_s, \phi_s)$ to the variables $(\theta_m, \omega_m, \phi_m, \theta, \omega, \phi)$ is a *homeomorphism*. We can solve six-dimensional differential equations of the master system and the difference dynamics, i.e., Eqs. (2) and (4) whose dynamics is identical to that of the original set of equations.

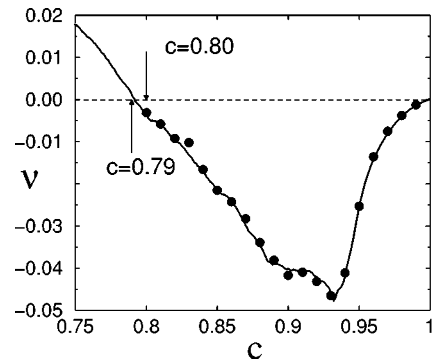

FIG. 1. The TLE ν is plotted against the coupling strength c (solid line). The filled circles represent the average decay rates of η vs time for which c is in the negative TLE regime, which well approximates the TLE [Fig. 2(d)].

In order to compare the conventional method with our scheme, we first calculate the transverse Lyapunov exponent (TLE) by studying the linearized equations of Eq. (4). Figure 1 shows that the presence of the parameter regime, i.e., $0.795 < c < 1.0$, in which the TLE is slightly negative. The TLE changes sign near $c = 0.795$. So we choose two parameter values, i.e., $c = 0.79$ in which the TLE is slightly positive and $c = 0.80$ in which the TLE is slightly negative, as our test values. Then, we calculate the time series of η for these two parameter values both in the conventional method and in our scheme to compare the results.

Figure 2 shows the striking difference between the conventional method and our scheme. According to the conventional calculations [Figs. 2(a) and (b)], the measure η abruptly goes to zero as the difference becomes less than 15 digits, whether the TLE is slightly positive for $c = 0.79$ [Fig. 2(a)] or slightly negative for $c = 0.80$. [See Fig. 2(b).] On the

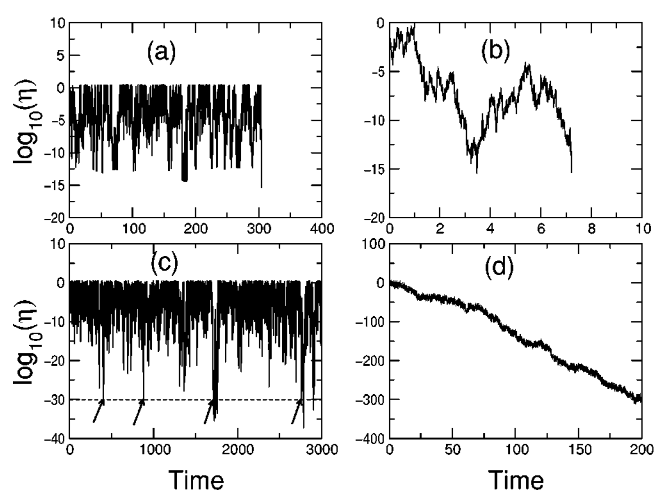

FIG. 2. Time series of η in log scale. The figures show the striking difference between results from the conventional calculations (a) and (b) and the calculation from our scheme (c) and (d). The coupling strength for (a) and (c) is $c = 0.79$ and (b) and (d) is $c = 0.80$.

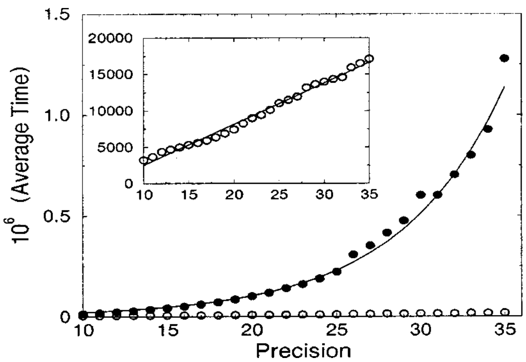

FIG. 3. The average time taken for synchronization vs the precision of calculations for $c=0.79$ (filled circle) and $c=0.80$ (open circle, inset). The solid line is the best fitted exponential curve.

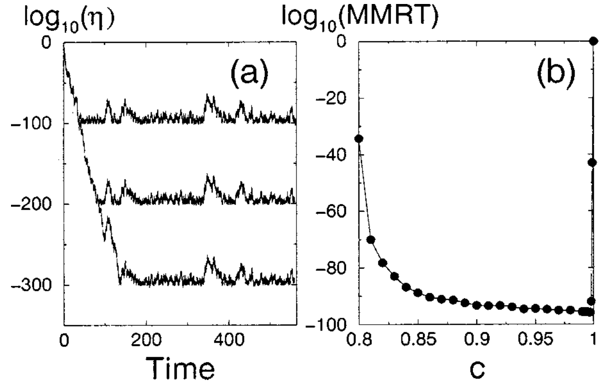

FIG. 4. The effects of added noise with varying amplitude in log scale. (a) The amplitudes of added noise are of the order of 10^{-100}, 10^{-200}, and 10^{-300}, respectively, from top to bottom. Once the dynamics reaches the order of noise level, the dynamics occasionally rebounds up to the order of 10^{48} times in this system. (b) MMRT when a noise of an amplitude of 10^{-100} is added for $0.80<c<1.0$.

other hand, the calculations from our scheme [Figs. 2(c) and 2(d)] show that the dynamics can be studied until the measure η reaches up to the order of 10^{-308} [14]. Figure 2(c) shows the time series of η lasts much longer than in Fig. 2(a), until it accidentally becomes less than 10^{-308} for the coupling strength $c=0.79$. This indicates the synchronization observed in Fig. 2(a) is an accidental one. On the contrary, the η in Fig. 2(d) shows a consistent decreasing tendency, i.e., exponential decay, as time goes on. It can be understood that the observed synchronization in Fig. 2(b), by the conventional method, is a genuine one. Moreover, in Fig. 2(d), we can estimate the TLE by measuring the average slope of the decay. The filled circles in Fig. 1. show the average decay rate (the slope of ν versus time), which well approximates the TLE, as c varies.

As we mentioned previously, Longa *et al.*'s criterion give good estimates for synchronization of coupled maps. However, the calculation for coupled chaotic flows by using symbolic languages like Mathematica or Maple packages takes an extremely long time. So the calculation of the average time taken for synchronization versus precision is practically infeasible when we use symbolic packages. (It can be calculated with the algorithm in Ref. [16] within single and double precision calculations). In our scheme, on the other hand, it can be easily estimated. In Fig. 2(c), we can measure the time intervals of two adjacently located instant of time at which $\log_{10}(\eta)$ reaches to the given digit [15] in time series of $\log_{10}(\eta)$, e.g., the arrows in Fig. 2(c) point $\log_{10}(\eta)$ reaches the dotted line, i.e., 30th digit. By taking an average of these time intervals, we can easily obtain the relation between average synchronization time and precision. Figure 3 shows the results of the *exponential* relation for $c=0.79$ and the *linear* relation for $c=0.80$. So, according to Longa *et al.*'s criterion, *we can conclude that the observed synchronization for* $0.80<c<1.0$ *in BBS's pendula is genuine synchronization.*

So far, we have illustrated that our scheme has various advantages over the conventional method for distinguishing the synchronizations. Even though we have concluded that coupled chaotic pendula can actually be synchronized for certain coupling strength regimes, the question that small additive noise might lead large intermittent bursts still remains. This question motivates us to study the numerical effects of small additive noise on the system, since BBS effectively added a small noise in their study. In order to investigate, we add noises of the order of 10^{-100}, 10^{-200}, and 10^{-300} to the dynamics of difference motion of coupled chaotic pendula for the coupling strength $c=0.80$. Remember that actual synchronization is possible for $c=0.80$. The typical behavior of the system is shown in Fig. 4(a). Once the dynamics reaches the order of noise level, the dynamics occasionally rebounds up to the order of 10^{64} times. This indicates that the system could show intermittent bursts, no matter how small a noise is added. So we investigate the maximum bouncing height versus the coupling strength c by running the calculation up to 10^7 in a normalized time unit when the added noise is of the amplitude of 10^{-100}. The result is shown in Fig. 4(b) that represents the magnitude of the maximum rebounding of the trajectory (MMRT) for a given noise amplitude, e.g., the applied noise of amplitude 10^{-100} can lead to MMRT of the order 10^{-34}, for $c=0.8$. [See Fig. 4(b)]. This is the crucial difference between coupled pendula and coupled logistic maps in which synchronization is achieved, regardless of additive noise in certain coupling strength regime. In case of coupled chaotic pendula, however, small added noises can have a big impact on their dynamics. This is the difficulty one can have in treating BBS type of systems.

Finally, we would like to comment that our scheme can be applied not only to coupled pendula, but also to generic coupled chaotic oscillators like Lorenz, Rössler, Duffing, forced Brusselator, etc. As an example, we show the result of coupled Duffing oscillators in Fig. 5.

In conclusion, we have clarified the controversies concerned with BBS's system by proposing a scheme to calculate the dynamics of coupled chaotic flows. The scheme provides a very simple way to test whether the observed synchronization is genuine or accidental. Using this scheme we are also able to study the effects of a small added noise on synchronization. Our results show that *there exists a certain parameter interval in which the coupled pendula system is actually synchronized.* This conclusion is based on Longa

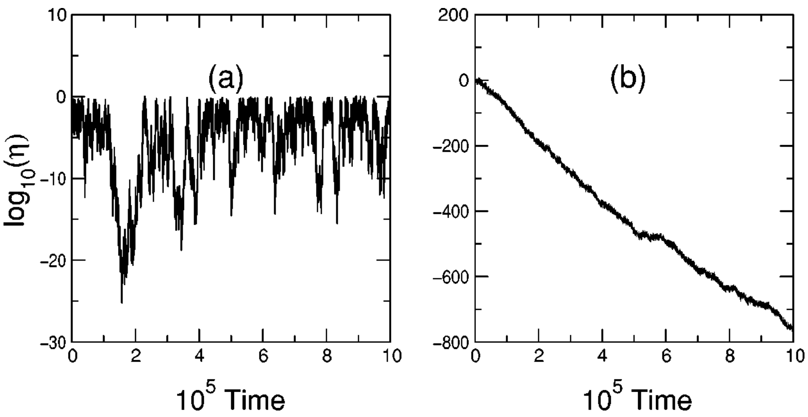

FIG. 5. Time series of η in log scale for unidirectionally coupled Duffing oscillators that are given by the following equations. $\ddot{x}+0.25\dot{x}-x+x^3=0.3\cos t$ and $\ddot{x}'+0.25\dot{x}'-x'+x'^3=0.3\cos t+c(x-x')$, where c is the coupling strength. (a) For $c=-0.755$, $\log_{10}\eta$ bursts intermittently. (b) For $c=-0.770$, $\log_{10}\eta$ decreases consistently.

et al.'s criterion. It is consistent with the result of TLE calculation. Even though our conclusion may sound against BBS's claim, we have also observed that BBS's system is very sensitive to small additive noises, i.e., an extremely small noise can lead to the very large bursts of the system. Therefore, in a practical sense, *intermittent synchronization* is a very reasonable way to define the BBS-type systems. Actually in a real environment, we may observe only *intermittent synchronization* in those systems as was reported by BBS [17]. From this study, we have also found that Longa *et al.*'s criterion is a very simple and effective way to distinguish the genuineness of the observed synchronization in coupled chaotic flows, as well as in coupled maps.

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